MATH2050C Assignment 6

Deadline: Feb 18, 2025.

Hand in: 3.4 no. 4b, 7ad; 3.5 no. 5. Suppl. Problem no 5.

Section 3.4 no. 4, 6, 7ad, 8, 9, 11.

Supplementary Problems

1. Let $\{x_n\}$ be a positive sequence such that $a = \lim_{n\to\infty} x_{n+1}/x_n$ exists. Show that $\lim_{n\to\infty} x_n^{1/n}$ exists and is equal to a. Hint: Write

$$x_n = \frac{x_n}{x_{n-1}} \frac{x_{n-1}}{x_{n-2}} \cdots \frac{x_2}{x_1} x_1 \ .$$

   Also use Theorem 5.3.

2. Show that $\lim_{n\to\infty} \dfrac{n}{(n!)^{1/n}} = e$.

3. The concept of a sequence extends naturally to points in $\mathbb{R}^N$. Taking $N = 2$ as a typical case, a sequence of ordered pairs, $\{\mathbf{a}_n\}, \mathbf{a}_n = (x_n, y_n)$, is said to be convergent to $\mathbf{a}$ if, for each $\varepsilon > 0$, there is some n_0 such that

$$|\mathbf{a}_n - \mathbf{a}| < \varepsilon \ , \quad \forall n \geq n_0 \ .$$

   Here $|\mathbf{a}| = \sqrt{x^2 + y^2}$ for $\mathbf{a} = (x, y)$. Show that $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a}$ if and only if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

4. Bolzano-Weierstrass Theorem in $\mathbb{R}^N$ reads as, every bounded sequence in $\mathbb{R}^N$ has a convergent subsequence. Prove it. A sequence is bounded if $|\mathbf{a}_n| \leq M$, $\forall n$, for some number M.

5. Consider the sequence $\{x_n\}, x_n = \sum_{k=1}^n s_n 1/n^2$ where s_n is either 1 or -1. Show that $\{x_n\}$ is convergent.

6. Consider $x_n = (x_{n-1} + x_{n-2})/2, n \geq 3$ and $x_1 = 1, x_2 = 2$. Show that $\{x_n\}$ converges to $5/3$. Hint: To find the limit establish $x_{2n+1} = 1 + \frac{1}{2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{2n-1}}$. and $x_{2n} = x_{2n-1} + \frac{1}{2^{2n-2}}$.

Bolzano-Weierstrass Theorem

Recall the Nested Interval Property in Assignment 2.

Theorem 6.1 (Nested Interval Property). Let $I_n = [a_n, b_n], n \leq 1$, be a sequence of closed intervals satisfying $I_{n+1} \subset I_n$. Then $\bigcap_n I_n = [a, b]$ where $a = \sup_n a_n$ and $b = \inf_n b_n$. When $b_n - a_n \to 0$ as $n \to \infty$, $a = b$ and $\bigcap_n I_n = \{a\}$.

Theorem 6.2 (Bolzano-Weierstrass Theorem). Every bounded sequence has at least one convergent subsequence.

Our proof is slightly different from the second proof in our text book.

Proof. Let $\{x_n\}$ be a bounded sequence. Fix a closed, bounded interval $I_0 = [-M, M]$ containing the sequence. We divide I_0 equally into two closed subintervals. One of these subintervals must contain infinitely many x_n's. Pick and call it I_1. Next, we divide I_1 equally into two closed subintervals and apply the same rule of selection to get I_2. Repeating this process, we end up with closed intervals $I_n, n \geq 1$, with the properties: For $n \geq 1$, (a) $I_{n+1} \subset I_n$, (b) the length of I_{n+1} is half that of I_n and so the length of I_n is equal to $M/2^{n-1}$ and (c) there are infinitely many x_n's sitting inside each I_n. By the Nested Interval Property $\cap_{n=1}^{\infty} I_n = \{z\}$. Now, we pick x_{n_1} from I_1 and then x_{n_2} from I_2 so that $n_2 > n_1$, which is possible since there are infinitely many x_n's in each I_n. Keep doing so we finally obtain a subsequence $\{x_{n_k}\}, x_{n_k} \in I_k$. From $a_k \leq x_{n_k} \leq b_k$ and $b_k - a_k \to 0$ we conclude $x_{n_k} \to z$ as $n \to \infty$.

A point z is called a limit point of the sequence $\{x_n\}$ if it is the limit of some subsequence of $\{x_n\}$. A bounded sequence has at least one limit point according to Bolzano-Weierstrass Theorem. This following theorem is the same as Theorem 3.4.9 in text book.

Theorem 6.3. A bounded sequence is convergent if all its convergent subsequences have the same limit.

Proof. Assume that there is only one limit point x. Suppose on the contrary that the sequence does not converge to x. We can find some $\varepsilon_0 > 0$ and $n_k \to \infty$ such that $|x_{n_k} - x| \geq \varepsilon_0$. Since $\{x_{n_k}\}$ is bounded, it contains a subsequence $\{x_{n_{k_j}}\}$ which converges to some y satisfying $|y - x| = \lim_{j \to \infty} |x_{n_{k_j}} - x| \geq \varepsilon_0$. Since every subsequence of a subsequence is a subsequence of the original sequence, $\{x_{n_{k_j}}\}$ is a subsequence of $\{x_n\}$. Thus y is a limit point different from x, contradicting our assumption that all convergent subsequences have the same limit.

On the other hand, if $x_n \to x$, for $\varepsilon > 0$, there is some n_0 such that $|x_n - x| < \varepsilon$ for all $n \geq n_0$. Let $\{y_k\}, y_k = x_{n_k}$, be a subsequence of $\{x_n\}$. Fix some $n_{k_0} \geq n_0$. Then we have $|y_k - x| = |x_{n_k} - x| < \varepsilon$ for all $n_k \geq n_{k_0}$, that is $y_k \to x$ too.

Let $\{x_n\}$ be a bounded sequence. For each $n \geq 1$, let

$$y_n = \sup_{k \geq n} x_k = \sup\{x_n, x_{n+1}, x_{n+2}, \cdots\} .$$

It is clear that $\{y_n\}$ is decreasing and bounded from below. By Monotone Convergence Theorem, its limit exists. We call it the limit superior of the sequence of $\{x_n\}$. In notation,

$$\overline{\lim}_{n \to \infty} x_n = \lim_{n \to \infty} y_n = \inf_{n \geq 1} \{y_n\} = \inf_{n \geq 1} \sup_{k \geq n} x_k .$$

Similarly, let

$$z_n = \inf_{k \geq n} x_k = \inf\{x_n, x_{n+1}, x_{n+2}, \cdots\} .$$

It is clear that $\{z_n\}$ is increasing and bounded from above. By Monotone Convergence Theorem, its limit exists. We call it the limit inferior of the sequence of $\{x_n\}$. In notation,

$$\underline{\lim}_{n\to\infty} x_n = \lim_{n\to\infty} z_n = \sup_{n\geq 1}\{z_n\} = \sup_{n\geq 1}\inf_{k\geq n} x_k .$$

Theorem 6.4. For a bounded sequence, its supremum is its largest limit point and its infimum the smallest limit point.

The following proof may be skipped in a first reading.

Proof *. Let b be the supremum of all limit points of $\{x_n\}$ and $a = \limsup_n x_n$. We want to show $a = b$. First, we claim that a is itself a limit point. Hence $a \leq b$. To do this we need to produce a subsequence convergence to a. For $\varepsilon = 1$, there is some n_0 such that $|y_n - a| < 1$ for all $n \geq n_0$. In particular, $|y_{n_0} - a| < 1$. Since $y_n = \sup\{x_n, x_{n+1}, x_{n+2}, \cdots,\}$, for the same $\varepsilon = 1$, there is some $m_0 \geq n_0$ such that $|x_{m_0} - y_{n_0}| < 1$. Next, by the same reasoning, for $\varepsilon = 1/2$, there is some $n_1 > n_0$ such that $|y_{n_1} - a| < 1/2$ and $m_1 \geq n_1$ such that $|y_{n_1} - x_{m_1}| < 1/2$. Continuing this, we obtain y_{n_k} and x_{m_k} where n_k and m_k are strictly increasing which satisfy $|y_{n_k} - a|, |y_{n_k} - x_{m_k}| < 1/k$. Therefore,

$$|x_{m_k} - a| \leq |x_{m_k} - y_{n_k}| + |y_{n_k} - a| < \frac{1}{k} + \frac{1}{k} = \frac{2}{k} .$$

Letting $k \to \infty$, by Squeeze Theorem we conclude $\lim_{k\to\infty} x_{m_k} = a$, done.

On the other hand, to show $b \leq a$ it suffices to show $c \leq a$ for any limit point c. Let $c = \lim_{n_k\to\infty} x_{n_k}$ be such a limit point. For $\varepsilon > 0$, there is some n_{k_0} such that $c - \varepsilon < x_{n_k}$ for all $n_k \geq n_{k_0}$. As $x_k \leq z_k$ for all k, we have $c - \varepsilon \leq x_{n_k} \leq z_{n_k}$. Letting $k \to \infty$, $a = \lim_{n_k\to\infty} z_{n_k} \geq c - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $a \geq c$. Taking sup over c, we get $a \geq b$.

Now it is easy to show

Theorem 6.5. Let $\{x_n\}$ be a bounded sequence. Then

1. $\underline{\lim}_{n\to\infty} x_n \leq \overline{\lim}_{n\to\infty} x_n$,
2. $\{x_n\}$ is convergent iff $\underline{\lim}_{n\to\infty} x_n = \overline{\lim}_{n\to\infty} x_n$. When this holds, $\lim_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n$.

The limit superior and limit inferior will become important in the future, but not it suffices to know its definitions.

Cauchy Convergence Criterion

The Cauchy convergence criterion is the most general criterion for convergence. It works for non-monotone sequences.

A sequence $\{x_n\}$ is a Cauchy sequence if for every $\varepsilon > 0$, there is some n_0 such that $|x_m - x_n| < \varepsilon$ for all $m, n \geq n_0$.

Proposition 6.6 Every convergent sequence is a Cauchy sequence.

Proof Let $x_n \to x$. For $\varepsilon > 0$, there is some n_0 such that $|x_n - x| < \varepsilon/2, \forall n \geq n_0$. Therefore, for $m, n \geq n_0$,

$$|x_m - x_n| = |x_m - x + x - x_n| \leq |x_m - x| + |x - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

so $\{x_n\}$ is a Cauchy sequence.

Theorem 6.7 (Cauchy Convergence Theorem) Every Cauchy sequence converges.

Proof Let $\{x_n\}$ be a Cauchy sequence. First we show that it is bounded. For $\varepsilon = 1$, there is some n_0 such that $|x_m - x_n| < 1$ for $m, n \geq n_0$. In particular, $|x_n - x_{n_0}| < 1$ so $|x_n| \leq |x_n - x_{n_0}| + |x_{n_0}| < 1 + |x_{n_0}|, n \geq n_0$. Therefore, $|x_n| \leq \max\{|x_1|, \cdots, |x_{n_0-1}|, 1 + |x_{n_0}|\}$ for all n. Now we can apply Bolzano-Weierstrass Theorem to extract a convergent subsequence $\{x_{n_k}\}, x_{n_j} \to x$, from $\{x_n\}$. For $\varepsilon > 0, |x_{n_k} - x| < \varepsilon/2, \forall n_k \geq n_{k_0}$ for some n_{k_0}. On the other hand, $\{x_n\}$ is Cauchy, for $\varepsilon > 0$, there is some n_0 such that $|x_n - x_m| < \varepsilon/2$ for $m, n \geq n_0$. Fix some $n_k \geq n_0, n_{k_0}$, we have

$$|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

so $x_n \to x$.